Topology Vol. 5, pp. 343–346. Pergamon Press, 1966. Printed in Great Britain

AN UNSTABLE ADAMS SPECTRAL SEQUENCE

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(*Received* 18 *March* 1966)

§1. INTRODUCTION

USING THE lower central series of a semisimplicial group, Curtis [4] has defined for each space X a spectral sequence whose E^1-term depends only on H_*X and which, for X simply connected, converges to π_*X. The object of this note is to define (in §2) a *mod-p version of Curtis's spectral sequence* and to show that

(i) *the* E^1*-term is a* Z_p*-module which depends only on* $H_*(X; Z_p)$. (§3)

(ii) *if* X *is simply connected and has finitely generated homotopy groups, then the spectral sequence converges in the same sense as the Adams spectral sequence* [1] *to a quotient of* π_*X. (§4)

This mod-p spectral sequence seems to be a good candidate for an *Unstable Adams spectral sequence* since [2], 2.6, it coincides in the stable range (after a minor reindexing) with the Adams spectral sequence.

The results presented in this note are contained in the author's thesis presented to the Massachusetts Institute of Technology. The author wishes to thank Professor D. M. Kan for his advice and helpful criticism during the preparation of this work.

§2. THE SPECTRAL SEQUENCE

2.1. The lower p-central series

Let G be a group and p a prime. The *lower p-central series* of G is [8] the filtration

$$G = \Gamma_1 G \supseteq \Gamma_2 G \supseteq \dots \supseteq \Gamma_r G \supseteq \dots,$$

where $\Gamma_r G$ is the subgroup generated by all elements

$$[a_1, \dots, a_k]^{p^i}$$

for which $k \geqq 1$, $kp^i \geqq r$, and each $a_j \in G$. The symbol $[\ \ , \dots, \ \ ]$ denotes the simple commutator $[\dots[\ \ ,\ \ ], \dots],\ \ ]$.

† The author was partially supported by a National Science Foundation Fellowship.

2.2. The spectral sequence

If X is a connected semi-simplicial complex with base point, let GX be its loop group complex [6]. Then GX is a free group complex with $\pi_q GX = \pi_{q+1} X$. We now denote by $\{E^t X\}$ the sepctral sequence derived from the homotopy exact couple of the filtered group complex GX,

$$GX = \Gamma_1 GX \supseteq \Gamma_2 GX \supseteq \ldots \supseteq \Gamma_r GX \supseteq \ldots .$$

2.3. A generalization

As in [4], 1.6, the above spectral sequence can be generalized to the case of homotopy classes of maps of $S^{q+1} Y$ into X, $q \geqq 1$. The obvious generalizations of the results of §3 then hold. For convergence one requires that Y be finite dimensional, that X be simply connected, and that both $H_n Y$ and $\pi_n X$ be finitely generated for all n.

§3. PROPERTIES OF $E^1 X$

Let X be a connected semisimplicial complex and $\{E^t X\}$ its mod-p spectral sequence.

THEOREM 3.1. *$E^1 X$ is a Z_p-module and depends only on $H_*(X; Z_p)$.*

Proof. We have $GX/\Gamma_2 GX \approx Z_p \otimes GX/[GX, GX]$; thus [6],

(3.2) $$\pi_q(GX/\Gamma_2 GX) \approx H_{q+1}(X; Z_p).$$

The group homotopy type of the Z_p-module complex $GX/\Gamma_2 GX$ is, therefore, (see [5]) determined by $H_*(X; Z_p)$.

In order to prove, for $r > 1$, that $\pi_*(\Gamma_r GX/\Gamma_{r+1} GX)$ depends only on $H_*(X; Z_p)$, we recall the definition of the free restricted Lie algebra on a Z_p-module M. Let TM be the tensor algebra $TM = \sum_{r \geqq 0} M^r$, where $M^r = M \otimes \ldots \otimes M$ r-times. For $a, b \in TM$, define $[a, b] = ab - ba$ and $a^{[p]} = a^p$; then the *free restricted Lie algebra LM* on M is the smallest sub Z_p-module of TM containing M and closed under the operations [,] and $(\ )^{[p]}$. Put $L_r M = LM \cap M^r$ so that $LM = \sum_{r \geqq 1} L_r M$. For each r, $L_r M$ is a functor of M. A result of Zassenhaus [8], §2, is

PROPOSITION 3.3. *If G is a free group, there is for each r a natural isomorphism*

$$\Gamma_r G/\Gamma_{r+1} G \approx L_r(G/\Gamma_2 G).$$

Applying this to GX, we have

PROPOSITION 3.4. $E^1 X \approx \pi_* L(GX/\Gamma_2 GX)$.

From 3.2, 3.4, and Dold's lemma [5], Theorem 3.1 now follows immediately.

3.5. Presentation of $E^1 X$ in terms of $H_*(X; Z_p)$

It turns out that $E^1 X$ is simpler than the corresponding term in Curtis's spectral sequence. There follows a presentation of $E^1 X$ for $p = 2$ (A. K. Bousfield, unpublished). A similar but more complicated presentation exists for p odd.

PROPOSITION. *Let X be simply connected, $p = 2$; then there is a natural isomorphism*

$$(E^1X)_{j+1} \approx \sum_{i \geq 0} \{L^G(S^{-1}H_*(X; Z_2))\}_{i+1} \otimes \pi_j L(AS_i),$$

where

(i) *$S^{-1}H_*(X; Z_2)$ is $H_*(X; Z_2)$ with gradation reduced by* 1.

(ii) *L^G is the free restricted graded Lie algebra functor* [7], §6.

(iii) *the groups $\pi_* L(AS_i)$ are as in* [2], 5.4.

§4. CONVERGENCE OF THE SPECTRAL SEQUENCE

Denote by $\pi_*(X; p)$ the quotient of $\pi_* X$ by the subgroup of elements of finite order prime to p.

THEOREM 4.1. *If X is simply connected and has finitely generated homotopy groups; then $\{E^r X\}$ is weakly convergent* [3, XV. 2], *and $E^\infty X$ is the graded group associated with a filtration of $\pi_*(X; p)$.*

Proof. It suffices to show that, for each r,

(4.2.) $\quad u \in \mathrm{Im}[\pi_* \Gamma_s GX \to \pi_* \Gamma_r GX]$ *for all $s \geqq r$ if and only if u is of finite order prime to p.*

Since each $\pi_*(\Gamma_r GX/\Gamma_{r+1} GX)$ is a Z_p-module, the "if" part of 4.2 is obvious. Now, in view of 3.1 and the assumptions on X, the groups $\pi_q \Gamma_r GX$ are all finitely generated. Therefore, an element of $\pi_* \Gamma_r GX$ is of finite order prime to p if it is infinitely divisible by p. By [8], 11, there is a semisimplicial map $\xi(r) : \Gamma_r GX \to \Gamma_{pr} GX$ sending $a \to a^p$. Now 4.2 follows easily from

LEMMA (4.3). *For each q there is an n_q such that $pr \geqq n_q$ implies*

$$\xi_*(r) : \pi_q \Gamma_r GX \to \pi_q \Gamma_{pr} GX$$

is an isomorphism.

Proof. Filter $\Gamma_s GX$, for each s, by

$$\Gamma_s GX = \Gamma_{s,1} GX \supseteq \Gamma_{s,2} GX \supseteq \dots \supseteq \Gamma_{s,m} GX \supseteq \dots,$$

where, for any group G, $\Gamma_{s,m} G$ is the subgroup generated by all elements

$$[a_1, \dots, a_k]^{p^i}$$

for which $k \geqq m$, $kp^i \geqq s$, and each $a_j \in G$. If $m \geqq s$, $\Gamma_{s,m} G$ is the m-th term in the *lower central series* of G. Now by [8], 15, *$\xi(r)$ induces isomorphisms*

$$\Gamma_{r,m} GX/\Gamma_{r,m+1} GX \approx \Gamma_{pr,m} GX/\Gamma_{pr,m+1} GX$$

for $m < pr$. Furthermore, by a theorem of Curtis [4], 1.3, for each q there is an N such that $m \geqq N$ implies $\Gamma_{m,m} GX$ is q-connected. Put $n_q = N$; *then $\Gamma_{r,pr} GX = \Gamma_{pr,pr} GX$ is q-connected for $pr \geqq n_q$.* Iterated application of the five-lemma now demonstrates 4.3.

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